

Analytic properties of unitarization schemes

O. V. SELUGIN ^{*})

BLTPh, Joint Institute for Nuclear Research, Dubna, Russia

J.-R. CUDELL [‡])

Institut de Physique, Bât. B5a, Université de Liège, Sart Tilman, B4000 Liège, Belgium

The analytic properties of the elastic hadron scattering amplitude are examined in the impact parameter representation at high energies. Different unitarization procedures and the corresponding non-linear equations are presented. Several unitarisation schemes are presented. They lead to almost identical results at the LHC.

PACS: 62.20

Key words: high energies, unitarity, non-linear equations, saturation

1 Introduction

Saturation is now a very popular term, but it has a very wide meaning. It includes shadowing and antishadowing processes, the saturation of the Froissart-Martin bound [1], gluon merging at small x and geometrical scaling, unitarization effects, etc. In the S -matrix language, saturation means that we reach the maximum possible scattering. This happens for a specific value of the impact parameter, and is directly connected with the unitary property of the scattering amplitude. It is that last meaning that we shall use in our work.

The most important results on the energy dependence of diffractive hadronic scattering were obtained from first principles (analyticity, unitarity and Lorentz invariance), which lead to specific analytic forms for the scattering amplitude as a function of its kinematical parameters – s , t , and u . Analytic S -matrix theory relates the high-energy behaviour of hadronic scattering to the singularities of the scattering amplitude in the complex angular momentum plane. One of the important theorems is the Froissart-Martin bound, which states that the high-energy cross section for the scattering of hadrons is limited by

$$\sigma_{tot}^{max} = \frac{2\pi}{\mu^2} \log^2\left(\frac{s}{s_0}\right), \quad (1)$$

where s_0 is a scale factor and μ the lightest hadron mass (*i.e.* the pion mass). As the coefficient in front of the logarithm is very large, “saturation of the Froissart-Martin bound” usually refers to an energy dependence of the total cross section rising as $\log^2 s$ rather than to the actual limit for the total cross section.

Experimental data reveal that total cross sections grow with energy. This means that the leading contribution in the high-energy limit is given by the rightmost

^{*}) e-mail: selugin@theor.jinr.ru

[‡]) e-mail: JR.Cudell@ulg.ac.be

singularity in the complex- j plane, the pomeron, with intercept exceeding unity. In the framework of perturbative QCD, the intercept is expected to exceed unity by an amount proportional to α_s [2]. At leading-log s , one obtains a rightmost singularity at $J - 1 = 12 \log 2(\alpha_s/\pi)$. Such a singularity – the hard pomeron – seems to be present in inelastic diffractive processes [3] and may be present in soft scattering, where it appears as a simple pole with intercept $\alpha_H \approx 1.4$ for energies smaller than 100 GeV [4, 5]. In this case, the Froissart-Martin bound is soon violated, and the BDL-regime may appear at relatively low energies. The effect of saturation on the growth of the total cross section is however far from clear, because it involves processes at very small x and non-perturbative effects.

Saturation in the framework of perturbative QCD may be connected with the growth of the gluon density at small x , which must be bounded by non-linear effects. In principle, the gluon density may have a maximum which will be reached before the BDL (see *e.g.* [6]). In fact, the non-linear corrections added to the BFKL equation lead to a saturation regime at relatively large impact parameters, and are not directly related with the BDL, which first occurs at small impact parameters. One must note that, even in perturbative QCD, the description of saturation at large impact parameters is problematic because of non-perturbative effects, connected with confinement [7].

2 Unitarity: schemes and equations

Unitarity of the scattering matrix

$$SS^+ = 1. \quad (2)$$

is most suitably studied in the impact parameter representation [8] as it is equivalent (at high energy) to a decomposition in partial-wave amplitudes.

We thus take the scattering amplitude in the impact parameter representation

$$T(s, t) = \int_0^\infty b db J_0(b\Delta) G(b, s). \quad (3)$$

with

$$ImG(b, s) \leq 1. \quad (4)$$

and

$$ImG(s, b) = [Im(G(s, b))]^2 + [ReG(s, b)]^2. \quad (5)$$

As energy grows, the scattering amplitude in the impact parameter representation can exceed the unitarity bound for a range of impact parameters $b < b_i$. To restore unitarity, there are several different schemes. Two of them are based on the solution of the unitarity equation [9]. The first and most popular scheme seems to be the eikonal representation.

$$T(s, t) = i \int_0^\infty b db J_0(bq) (1 - \exp(-\chi(s, b))). \quad (6)$$

where the eikonal function $\chi(s, b)$, which in the non-relativistic regime is equal to the scattering potential, is often taken as the Born scattering amplitude.

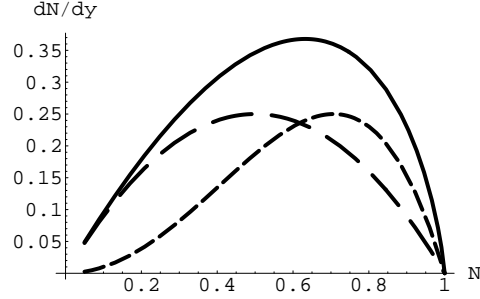


Fig. 1. The derivative of the non-linear equations corresponding to the eikonal (hard line), the rescaled U -matrix (dashed line) and the hyperbolic tangent (long dashed line) unitarization schemes.

The second unitarization scheme which we shall consider is the U -matrix scheme [10, 11], which we rescale so that the unitarized amplitude becomes asymptotically i as $s \rightarrow \infty$:

$$T(s, t) = \int_0^\infty b db J_0(bq) \left(\frac{\chi(s, b)}{1 - i\chi(s, b)} \right). \quad (7)$$

with $t = -q^2$. If we assume that the Born scattering amplitude factorises as $f(b)s^{1+\Delta}$, we can rewrite this scheme as the solution of the non-linear equation

$$\frac{dN}{dy} = \Delta N[1 - N]. \quad (8)$$

We find that eikonal representation also corresponds the non-linear equation like above [12].

$$\frac{dN}{dy} = -\Delta \log(1 - N)(1 - N). \quad (9)$$

We can also obtain the above schemes by assuming that $dN/dN_{Born} = D(N)$, in which case

$$\frac{dN}{dy} = \frac{dN_{Born}}{dy} D(N). \quad (10)$$

We can also easily obtain new unitarization schemes [13], such as

$$T(s, t) = \int_0^\infty b db J_0(bq) \tanh[\chi(s, b)]. \quad (11)$$

which corresponds to the non-linear equation in the form

$$\frac{dN}{dy} = \frac{dN_{Born}}{dy} (1 - N^2). \quad (12)$$

At first sight, these three schemes are quite different. However we can represent them in one form

$$T(s, t) = i \int_0^\infty b db J_0(bq) (1 - F(\chi(s, b))). \quad (13)$$

In order for Eq. (13) to produce a unitary amplitude, it is sufficient to impose the following properties on F :

- a) it must be an analytic function of its variable s and b , and satisfy crossing symmetry,
- b) it must not exceed unity,
- c) when $\chi(s, b) \rightarrow 0$, $F \rightarrow 1$,
- d) when $\chi(s, b) \rightarrow \infty$, $F \rightarrow 0$,
- e) at large b , it must fall faster than a power.

The functions $F(s, b)$ of the above unitarization schemes satisfy such conditions. For the eikonal it has the standard form

$$F(s, b) = \exp[-\chi(s, b)], \quad (14)$$

for the rescaled U -matrix

$$F(s, b) = \frac{1}{1 + \chi(s, b)}, \quad (15)$$

and for hyperbolic tangent representation

$$F(s, b) = \frac{2 \exp(-2\chi(s, b))}{1 + \exp(-2\chi(s, b))}, \quad (16)$$

The differences between these functions is connected with different form of the saturation regime.

In the following, we shall assume that the scattering amplitude in the impact parameter representation can be factorised $\chi(s, b) \sim h(s)f(b)$, and that the energy dependence of $h(s)$ is a power

$$h(s) \sim s^\Delta. \quad (17)$$

We find that the energy dependence of the imaginary part of the amplitude and hence of the total cross section depends on the form of $f(b)$, *i.e.* on the s and t dependence of the slope of the elastic scattering amplitude.

In Fig. 1 the process of saturation of the unitarity bound in the case of the three unitarization schemes is presented. It is clear that despite the slight difference in the growth the overlapping function with energy, the profile at the LHC will be similar for all these three unitarization schemes (see Fig. 2).

The overlapping function is complex, because it must be crossing symmetric. In that case, s can be replaced by $s \exp(-i \pi/2)$. At very low energies, before the first inelastic channel opens, the total cross section is equal to the elastic cross section and the overlapping function has a large real part. When the energy grows

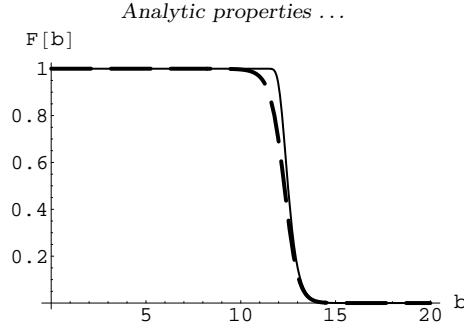


Fig. 2. $G(s_{LHC}, b)$, for the eikonal (hard line) and the rescaled U -matrix (long-dashed line) forms at super high energies.

and the overlapping function reaches the Black Disk Limit (BDL) (see, for example [14, 15]), the elastic cross section will be one half of the total cross section. Then the real part of the overlapping function disappears at small impact parameters. This has a small impact on the behavior of the total cross section, as shown in Fig. 3, where we compare the values of σ_{tot} in the case of a purely imaginary eikonal and of a complex eikonal.

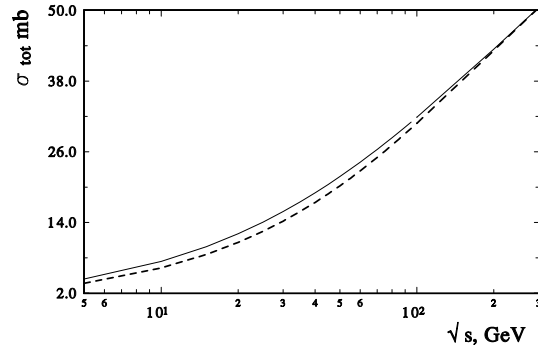


Fig. 3. The total cross sections of proton-proton scattering are calculated for a purely imaginary eikonal (plain line) and for a complex eikonal (dashed line).

3 Conclusion

We have shown that the impact parameter is the natural variable to study unitarity of the scattering amplitude. In particular, the unitarity bound leads to the Black Disk Limit at high energies. The corresponding saturation phenomena may be connected with the saturation of the partonic density of the interacting hadrons. As a step in that direction, we showed that the usual unitarization schemes are in one-to-one correspondence with nonlinear equations. Such an approach can be used

to build new unitarization schemes and may also shed some light on the physical processes underlying the saturation regime. In the presence of a hard Pomeron the saturation effects can change the behavior of some features of the cross sections already at LHC energies. In the unitarisation schemes considered here, non-linear effects appear before the BDL and lead to an acceptable growth of the total cross sections. Saturation then leads to a relative growth of the contribution of peripheral interactions, and to changes in energy dependence of the differential cross sections for moderate values of the momentum transfer. Such effects in the differential elastic cross section can be discovered at the LHC.

Acknowledgments We thank Jan Fisher for a stimulating discussion. O.V.S. acknowledges the support of FRNS (Belgium) for visits to the University of Liège where part of this work was done.

References

- [1] M. Froissart, Phys. Rev. **123** (1961) 1053; A. Martin, Nuovo Cimento **A 42** (1965) 930.
- [2] L.N. Lipatov, Sov. J. Nucl. Phys. **23** (1976) 338; E.A. Kuraev, L.N. Lipatov, and V.S. Fadin, Sov. Phys. JETP **45** (1977) 199; I.I. Balitsky and L.N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822.
- [3] S. Donnachie, G. Dosch, O. Nachtmann and P. Landshoff, "Pomeron Physics And QCD," Cambridge: Cambridge University Press (2002) (Cambridge monographs on particle physics, nuclear physics and cosmology. 19).
- [4] J.-R. Cudell, A. Lengyel, E. Martynov, O.V. Selyugin, Nucl. Phys. A **755**, p. 587-590 [hep-ph/0501288].
- [5] J.-R. Cudell, A. Lengyel, E. Martynov and O.V. Selyugin, Phys. Lett. **B 587** (2004) 78; Nucl. Phys. **A 755** (2006) 587 [hep-ph/0501288].
- [6] E. Levin, Nucl. Phys. **A 763** (2005) 140.
- [7] A. Kovner, Lectures on XLV Cracow School of Theoretical Physics, Zakopane, June (2005) , hep-ph/0508232.
- [8] K.A. Ter-Martirosyan, Sov. ZhETF Pisma **15** (1972) 519; A.B. Kaidalov, L.A. Ponomarev, K.A. Ter-Martirosyan, Sov. J. Part. Nucl. **44** (1986) 468.
- [9] V. Barone and E. Predazzi, in book "High-energy particle diffraction", Springer, New York, (2002).
- [10] A.A. Logunov, V.I. Savrin, N.E. Tyurin, and O.A. Khrustalev, Theor. Mat. Fiz. **6** (1971) 157.
- [11] V.I. Savrin, N.E. Tyurin, O.A. Chrustalev, Fiz. Elem. Chast. At Yadra **7** (1976) 21.
- [12] J.-R. Cudell, O.V. Selyugin, Nucl. Phys. (Proc. Suppl.) **B 146** (2005) 185 [hep-ph/0412338].
- [13] J.-R. Cudell, O.V. Selyugin, Czech. J. Phys. **55** (2005) A235.
- [14] P. Desgrolard, L.L. Jenkovszky, B.V. Struminsky, Yad. Fiz. **63** (2000) 962.
- [15] J. Bartels, E. Gotsman, E. Levin, M. Lublinsky and U. Maor, Phys. Lett. B **556** (2003) 114 [arXiv:hep-ph/0212284].